UDC 517.929.4 Вестник СПбГУ. Прикладная математика. Информатика... 2019. Т. 15. Вып. 4
MSC 74J55

Stability of weak solutions of parabolic systems

A. P. Zhabko[1], *A. I. Shindyapin*[2], *V. V. Provotorov*[3]

[1] St. Petersburg State University, 7–9, Universitetskaya nab., St. Petersburg, 199034, Russian Federation
[2] Eduardo Mondlane University, 1, Julius Nyerere av., 3453, Maputo, Mozambique
[3] Voronezh State University, 1, Universitetskaya pl., Voronezh, 394006, Russian Federation

For citation: Zhabko A. P., Shindyapin A. I., Provotorov V. V. Stability of weak solutions of parabolic systems with distributed parameters on the graph. *Vestnik of Saint Petersburg University. Applied Mathematics. Computer Science. Control Processes*, 2019, vol. 15, iss. 4, pp. 457–471. https://doi.org/10.21638/11702/spbu10.2019.404

The analysis of the behaviour of the evolutionary equation solution with unlimited time variable has been a subject of discussion in scientific circles for a long time. There are many practical reasons for this when the initial conditions of the equation are specified with a certain error: how the small changes in the initial conditions affect the behaviour of the solution for large values of the time. The paper uses the classical understanding of the stability of the solution of a differential equation or a system of equations that goes back to the works of A. M. Lyapunov: a solution is stable if it little changes under the small perturbations of the initial condition. In the work specified the stability conditions for the solution of an evolutionary parabolic system with distributed parameters on a graph describing the process of transfer of a continuous mass in a spatial network are indicated. The parabolic system is considered in the weak formulation: a weak solution of the system is a summable function that satisfies the integral form identity, which determines the variational formulation for the initial-boundary value problem. By going beyond the classical (smooth) solutions and addressing weak solutions of the problem the authors aim not only to describe more precisely the physical nature of the transfer processes (this takes on particular importance when studying the dynamics of multiphase media) but also to the path analysis processes in multidimensional network-like domains. The used approach is based on a priori estimates of the weak solution and the construction (the Fayedo—Galerkin method with a special basis — the system of eigenfunctions of the elliptic operator of a parabolic equation) of a weakly compact family of approximate solutions in the selected state space. The obtained results underlie the analysis of optimal control problems for differential systems with distributed parameters on a graph, which have interesting analogies with multiphase problems of multidimensional hydrodynamics.

Keywords: evolutionary system of parabolic type, distributed parameters on the graph, a weak solution, stability of a weak solutions.

1. Introduction. Today have a lot of the results on the mathematical theory of stability, however, as we know they are all overwhelmingly oriented on the ordinary differential equations and systems. In many applications because of the complexity of the mathematical models have to abandon the use of ordinary differential equations in behalf of considering the evolutionary equations with partial derivatives. Precisely this case is the object of the study in this work, in which represent the analysis of the stability of the weak solutions of the evolutionary systems with distributed parameters on the graph with unlimited growing of the temporary variable. By studying the relevant initial-boundary value problem, we to by beyond the scope a classical solutions and reduce to the weak solutions of the problem, reflecting more accurately the physical essence of appearance and processes (i. e. consider the initial-boundary value problem in weak formulation). This solutions reflects more exactly the physical essence of phenomena and processes. It is well known the choice of the class of weak solutions to be determined one way or the other functional space that is available to the researcher. This choice was made, first of all, the requirement of conservation of the existence theorem and the uniqueness theorem, if it is corresponds to the spirit of the studied phenomenon or process.

We are sufficiently detailed form here is the proof of the existence of weak solutions of initial-boundary value problem in distinct spaces (analogously reasoning shows in the works [1–4]). It is necessary to demonstrate the equivalent concept of Lyapunov stability of the undisturbed state differential system for differential equations with partial derivatives and show the ability to use known results [5, 6] in the studied case.

2. Notation, concepts and basic statement. We introduce the following concepts and symbols adopted in the works [3, 4]: Γ is the limited oriented geometric graph with edges γ, parameterized the segment $[0, 1]$; $\partial\Gamma$ and $J(\Gamma)$ are the many of boundary ζ and internally ξ nodes of graph respectively; Γ_0 is the join of all the edges of the graph Γ, does not contain the endpoints; $\Gamma_t = \Gamma_0 \times (0, t)$ ($\gamma_t = \gamma_0 \times (0, t)$), $\partial\Gamma_t = \partial\Gamma \times (0, t)$ ($t \in (0, T]$, $T < \infty$).

In all the course of the work make use of the Lebesgue integral along Γ or Γ_t:

$$\int\limits_{\Gamma} f(x)dx = \sum\limits_{\gamma} f(x)_{\gamma} dx \quad \text{or} \quad \int\limits_{\Gamma_t} f(x,t)dxdt = \sum\limits_{\gamma_t} f(x,t)_{\gamma} dxdt,$$

$f(\cdot)_{\gamma}$ is narrowing function $f(\cdot)$ on the edge γ.

We use classic space of functions: $C(\Gamma)$ is a space continuous functions on Γ; $L_p(\Gamma)$ ($p = 1, 2$) is a Banach space of measurable functions on Γ_0, integrable with degree of order p (similarly defined the space $L_p(\Gamma_T)$); $L_{2,1}(\Gamma_T)$ is the space of functions from $L_1(\Gamma_T)$ with norm defined by ratio $\|u\|_{L_{2,1}(\Gamma_T)} = \int\limits_0^T (\int\limits_{\Gamma} u^2 dx)^{\frac{1}{2}} dt$.

We also use analogues of Sobolev's spaces [7, 8]: $W_2^1(\Gamma)$ is the space of functions from $L_2(\Gamma)$, with generalized derivative of order 1 also from $L_2(\Gamma)$; $W_2^{1,0}(\Gamma_T)$ is the space of functions from $L_2(\Gamma_T)$ with generalized derivative of order 1 for x belonging to space $L_2(\Gamma_T)$ (similarly defined the space $W_2^1(\Gamma_T)$).

Relabel $V_2(\Gamma_T)$ the set of all functions $u(x,t) \in W_2^{1,0}(\Gamma_T)$ with finite norm

$$\|u\|_{2,\Gamma_T} \equiv \max_{0 \leqslant t \leqslant T} \|u(\cdot,t)\|_{L_2(\Gamma)} + \|u_x\|_{L_2(\Gamma_T)}. \tag{1}$$

In addition these functions are continuous on t in norm space $L_2(\Gamma)$, that is under $\Delta t \to 0$ $\|u(t + \Delta t) - u(t)\|_{L_2(\Gamma)} \to 0$ uniformly on $[0, T]$.

We will introduce the state space of parabolic system and the auxiliary space. To do this, consider the bilinear form

$$\ell(\mu, \nu) = \int\limits_{\Gamma} \left(a(x) \tfrac{d\mu(x)}{dx} \tfrac{d\nu(x)}{dx} + b(x)\mu(x)\nu(x) \right) dx$$

with a fixed measurable and limited on Γ_0 functions $a(x)$, $b(x)$ square integrable:

$$0 < a_* \leqslant a(x) \leqslant a^*, \quad |b(x)| \leqslant \beta, \quad x \in \Gamma. \tag{2}$$

Following approval is valid [9, p. 92].

Lemma 1. *Let the function* $u(x) \in W_2^1(\Gamma)$ *such that*

$$\ell(u,\nu) - \int\limits_{\Gamma} f(x)\nu(x)dx = 0$$

for any $\nu(x) \in W_2^1(\Gamma)$ *(*$f(x) \in L_2(\Gamma)$ *is the fixed function).*

Then for any edge $\gamma \subset \Gamma$ *the narrowing of the function* $a(x)_\gamma \frac{du(x)_\gamma}{dx}$ *continuously in the endpoints of the edge* γ.

Relabel $\Omega_a(\Gamma)$ the many of these functions $u(x)$ satisfying the ratios

$$\sum_{\gamma \in R(\xi)} a(1)_\gamma \frac{du(1)_\gamma}{dx} = \sum_{\gamma \in r(\xi)} a(0)_\gamma \frac{du(0)_\gamma}{dx}$$

in all nodes $\xi \in J(\Gamma)$ (in here $R(\xi)$ and $r(\xi)$ as the sets of the edges γ accordingly oriented "to node ξ" and "from node ξ") and $u(x)|_{\partial\Gamma} = 0$. The closing of the set $\Omega_a(\Gamma)$ in norm $W_2^1(\Gamma)$ relabel $W_0^1(a,\Gamma)$.

Let the next $\Omega_a(\Gamma_T)$ is the set of functions $u(x,t) \in V_2(\Gamma_T)$, whose traces are defined in sections of the domain Γ_T the plane $t = t_0$ ($t_0 \in [0,T]$) as a function of class $W_0^1(a,\Gamma)$ and satisfy a ratios

$$\sum_{\gamma \in R(\xi)} a(1)_\gamma \frac{\partial u(1,t)_\gamma}{\partial x} = \sum_{\gamma \in r(\xi)} a(0)_\gamma \frac{\partial u(0,t)_\gamma}{\partial x} \tag{3}$$

for all nodes $\xi \in J(\Gamma)$. The closing of the set $\Omega_a(\Gamma_T)$ in norm (1) relabel $V^{1,0}(a,\Gamma_T)$; it is clear that $V^{1,0}(a,\Gamma_T) \subset W_2^{1,0}(\Gamma_T)$.

Another the subspace of the space $W_2^{1,0}(\Gamma_T)$ is $W^{1,0}(a,\Gamma_T)$; $W^{1,0}(a,\Gamma_T)$ is the closing in norm $W_2^{1,0}(\Gamma_T)$ the set of differentiable on Γ_0 functions $u(x,t)$ satisfy a ratios (3) for all nodes $\xi \in J(\Gamma)$ and boundary condition $u(x,t)|_{\partial\Gamma} = 0$ for any $t \in [0,T]$ (the derivatives in the nodes are defined as the one-sided derivative). The space $W_2^{1,0}(\Gamma_T)$ is defined similarly.

Remark 1. The space $V^{1,0}(a,\Gamma_T)$ describes the states set of parabolic system, $W^1(a,\Gamma_T)$ is the auxiliary space. The distinction of elements space $V^{1,0}(a,\Gamma_T)$ from elements space $W^1(a,\Gamma_T)$ is the absence of elements of $W^1(a,\Gamma_T)$ continuity for the temporary variable t.

In spaces $W^1(a,\Gamma_T)$ and $V^{1,0}(a,\Gamma_T)$ consider the parabolic equation

$$\frac{\partial y(x,t)}{\partial t} - \frac{\partial}{\partial x}\left(a(x)\frac{\partial y(x,t)}{\partial x}\right) + b(x)y(x,t) = f(x,t), \tag{4}$$

represents a system of differential equations with distributed parameters on each edge γ of the graph Γ; $f(x,t) \in L_{2,1}(\Gamma_T)$. The state $y(x,t)$ ($x,t \in \overline{\Gamma}_T$) of the system (4) in the domain $\overline{\Gamma}_T$ is determined by a weak solution $y(x,t)$ of the equation (4), satisfying the initial and boundary conditions

$$y\,|_{t=0} = \varphi(x), \quad x \in \Gamma, \quad y\,|_{x \in \partial\Gamma_T} = 0, \tag{5}$$

where $\varphi(x) \in L_2(\Gamma)$. The suppositions about functions $a(x)$ and $b(x)$ to make mention above in (2). From $y(x,t) \in V^{1,0}(a,\Gamma_T)$ should be noted, that the map $y : [0,T] \to W_0^1(a,\Gamma) \subset L_2(\Gamma)$ is a continuous function, so that the first equality (5) makes sense and is be understood almost everywhere.

Remark 2. In the work considers the initial-boundary value problem (4), (5) with Dirichlet boundary conditions in the ratios (5), for other types of boundary conditions is given any comment.

We present the basic statements and the basic fragments of proofs of their first in auxiliary space $W^{1,0}(a,\Gamma_T)$, then in the space $V^{1,0}(a,\Gamma_T)$, full proofs given in the work [4].

Definition 1. *A weak solution of the initial-boundary value problem* (4), (5) class $W_2^{1,0}(\Gamma_T)$ *is the function* $y(x,t)\in W^{1,0}(a,\Gamma_T)$, *that satisfies an integral identity*

$$-\int\limits_{\Gamma_T} y(x,t)\frac{\partial\eta(x,t)}{\partial t}dxdt+\ell_T(y,\eta)=\int\limits_{\Gamma}\varphi(x)\eta(x,0)dx+\int\limits_{\Gamma_T} f(x,t)\eta(x,t)dxdt \tag{6}$$

for any function $\eta(x,t)\in W^1(a,\Gamma_T)$ *equal to zero under* $t=T$; $\ell_t(y,\eta)$ *is bilinear form defined by the ratio of*

$$\ell_t(y,\eta)=\int\limits_{\Gamma_t}\left(a(x)\frac{\partial y(x,t)}{\partial x}\frac{\partial\eta(x,t)}{\partial x}+b(x)y(x,t)\eta(x,t)\right)dxdt,\quad t\in(0,T].$$

Definition 2. *A weak solution of the initial-boundary value problem* (4), (5) class $W_2^{1,0}(\Gamma_T)$ *is the function* $y(x,t)\in V^{1,0}(a,\Gamma_T)$, *that satisfies an integral identity*

$$\int\limits_{\Gamma} y(x,t)\eta(x,t)dx-\int\limits_{\Gamma_t} y(x,t)\frac{\partial\eta(x,t)}{\partial t}dxdt+\ell_t(y,\eta)=$$
$$=\int\limits_{\Gamma}\varphi(x)\eta(x,0)dx+\int\limits_{\Gamma_t} f(x,t)\eta(x,t)dxdt$$

under any $t\in[0,T]$ *and for any function* $\eta(x,t)\in W^1(a,\Gamma_T)$.

In proving solvability the problem (4), (5) in the space $W^{1,0}(a,\Gamma_T)$ (and $V^{1,0}(a,\Gamma_T)$) is a special basis of space $W_0^1(a,\Gamma)$ — system of generalized eigenfunctions of boundary value problem on the eigenvalues (the spectral problem)

$$-\frac{d}{dx}\left(a(x)\frac{du(x)}{dx}\right)+b(x)u(x)=\lambda u(x),\;\; u(x)|_{\partial\Gamma}=0, \tag{7}$$

in class $W^1(a,\Gamma)$ [10, 11]. This problem consists in finding many such numbers (eigenvalues boundary value problem (7)), each of which corresponds to at least one nontrivial generalized solution $u(x)\in W_0^1(a,\Gamma)$ (the generalized eigenfunction), satisfies the integral identity

$$\ell(u,\eta)=\lambda(u,\eta) \tag{8}$$

for any function $\eta(x)\in W_0^1(a,\Gamma)$ (here and everywhere below through $(\cdot,\cdot)$ designated scalar product in $L_2(\Gamma)$ or $L_2(\Gamma_T)$). Install the necessary further properties of eigenvalues and generalized eigenfunctions of spectral problem (7). To do this, we will introduce in the space $W_0^1(a,\Gamma)$ of new scalar product

$$[u,v]=\int\limits_{\Gamma}\left(a(x)\frac{du(x)}{dx}\frac{dv(x)}{dx}+(\lambda_0+b(x))u(x)v(x)\right)dx=\ell(u,v)+\lambda_0(u,v),$$

where λ_0 defined by inequalities $\lambda_0>\beta$. Then given (2), correctly ratio

$$[u,v]\geqslant a_*\|\tfrac{du}{dx}\|^2+(\lambda_0-\beta)\|\tfrac{dv}{dx}\|^2\geqslant\alpha(\|\tfrac{du}{dx}\|^2+\|\tfrac{dv}{dx}\|^2),$$

$\alpha>0$ is fixed constant. The latter means equivalence of norms $\|u\|_{[\cdot,\cdot]}=\sqrt{[u,u]}$ induced by the scalar product $[\cdot,\cdot]$, and norm of space $W_0^1(a,\Gamma)$. The generalized eigenfunctions of spectral problem (7) satisfy the integral identity

$$[u,\eta]=(\lambda_0+\lambda)(u,\eta) \tag{9}$$

for any function $\eta(x)\in W_0^1(a,\Gamma)$.

In space $W_0^1(a,\Gamma)$ we define the operator B by using identities

$$[Bu,\eta]=(u,\eta)\quad \forall\eta(x)\in W_0^1(a,\Gamma),$$

then identity (9) is equivalent to

$$[Bu,\eta]=\tilde{\lambda}[u,\eta]\quad \forall\eta(x)\in W_0^1(a,\Gamma),$$

where $\tilde{\lambda}=1/(\lambda_0+\lambda)$. It is easy to verify that the operator B is completely, self-adjoint and positive. Whence it follows that the eigenvalues $\{\tilde{\lambda}_i\}$, $i\geqslant 1$, of its valid, positive and they can be considered numbered in descending order, taking into account their multiplicity, where $\tilde{\lambda}_i\to 0$ under $i\to\infty$. Note that the point $\tilde{\lambda}=0$ not its eigenvalue, because under $\tilde{\lambda}=0$ from the definition of the operator B derives $u=0$. The generalized eigenfunctions $\{u_i(x)\}_{i\geqslant 1}$, that correspond to their eigenvalues $\{\tilde{\lambda}_i\}_{i\geqslant 1}$, are real and mutually orthogonal: $[u_i,u_j]=0$ under $i\neq j$. In view of the foregoing, the system of generalized eigenfunctions $\{u_i(x)\}_{i\geqslant 1}$ form a basis in space $W_0^1(a,\Gamma)$, as $W_0^1(a,\Gamma)$ densely in $L_2(\Gamma)$, then $\{u_i(x)\}_{i\geqslant 1}$ is basic in $L_2(\Gamma)$ (you can take that the basis $\{u_i(x)\}_{i\geqslant 1}$ is orthonormalized in $L_2(\Gamma)$). Taking into account the connection $\lambda_i=-\lambda_0+1/\tilde{\lambda}_i$ of the eigenvalues λ_i of spectral problem (7) and the eigenvalues $\tilde{\lambda}_i$ of the operator B, as well as the coincidence by him corresponding generalized eigenfunctions, come to the following statement.

Lemma 2. *Let met supposition* (2). *Then the spectral problem* (7) *has the denumerable set of real eigenvalues* $\{\lambda_i\}_{i\geqslant 1}$ (*indexed in ascending order, taking into account their multiplicity*) *with limit point at infinity* (*the eigenvalues* λ_i *are positive, with the except ion of maybe finite number first*). *The system of generalized eigenfunctions* $\{u_i(x)\}_{i\geqslant 1}$ *form a basis in* $W_0^1(a,\Gamma)$ *and* $L_2(\Gamma)$, *orthonormalized in* $L_2(\Gamma)$ *and orthogonal in meaning of the scalar product* $[\cdot,\cdot]$.

Remark 3. If $0\leqslant b(x)\leqslant\beta$, as is usually the case in the applications, then all eigenvalues of the spectral problem (7) is positive. Indeed, this follows from the integral identities (8), when $u=\eta=u_i(x)$, $\lambda=\lambda_i$ and a chain of equalities

$$\ell(u_i,u_i)=\lambda_i(u_i,u_i)=\lambda_i\|u_i\|^2=\lambda_i$$

for $i=1,2,...$ (see also [9, p. 98]). The positiveness of the eigenvalues is the determining factor for establishing the stability condition and the possibility of the stability of evolutionary systems of parabolic equations with distributed parameters on the graph.

Theorem 1. *If any* $f(x)\in L_{2,1}(\Gamma_T)$, $\varphi(x)\in L_2(\Gamma)$ *and for any* $0<T<\infty$ *the initial-boundary value problem* (4), (5) *is weak solvability in space* $W^{1,0}(a,\Gamma_T)$.

P r o o f. In proving theorems construct the Faedo—Galerkin approximation on the basis $\{u_n(x)\}_{n\geqslant 1}$: the approximate solutions (N is fixed natural number) (4), (5) have the form $y^N(x,t)=\sum\limits_{i=1}^{N}c_i^N(t)u_i(x)$, where $c_i^N(t)$ is absolutely continuous on $[0,T]$ functions ($c_i'(t)\in L_2(0,T)$), defined from the system

$$\begin{gathered}\left(\frac{\partial y^N}{\partial t},u_i\right)+\int\limits_{\Gamma}\left(a(x)\frac{\partial y^N(x,t)}{\partial x}\frac{du_i(x)}{dx}+b(x)y^N(x,t)u_i(x)\right)dx=(f,u_i)\,,\\ c_i^N(0)=(\varphi,u_i),\ i=\overline{1,N}.\end{gathered}\tag{10}$$

Further reasoning based on a priori estimates of norm of weak solutions (4), (5) and construction the subsequence $\left\{y^{N_k}\right\}_{k\geqslant 1}$ of sequence $\left\{y^N\right\}_{N\geqslant 1}$, weakly converge to solution

$y(x,t) \in W^{1,0}(a,\Gamma_T)$ in a norm of $W^{1,0}(\Gamma_T)$ (weak compactness $\left\{y^{N_k}\right\}_{k\geqslant 1}$). Namely [4, 7], for approximate solutions $y^N(x,t)$ the inequality by

$$\|y^N\|_{2,\Gamma_t} \leqslant C(t)\left(\|y^N(x,0)\|_{L_2(\Gamma)} + 2\|f\|_{L_{2,1}(\Gamma_t)}\right),$$

is valid for any $t \in [0,T]$; here the function $C(t)$ is limited to $t \in [0,T]$ ($C(t) \leqslant C^*, C^* > 0$), not depend on N, is determined by the value T and permanent a^*, β. From this inequality, taking into account the ratio $c_i^N(0) = (\varphi, u_i)$ ($i = \overline{1,N}$) and by virtue of inequalities

$$\|y^N(x,0)\|_{L_2(\Gamma)} = \|\sum_{i=1}^{N}(\varphi,u_i)u_i(x)\| \leqslant \sqrt{\sum_{i=1}^{N}|(\varphi,u_i)|} \leqslant \|\varphi\|_{L_2(\Gamma)}$$

$\left(\|\cdot\| \text{ is Euclidean norm: } \|\omega\| = \sqrt{\sum_{i=1}^{N}\omega_i^2}\right)$, it should be

$$\|y^N\|_{2,\Gamma_t} \leqslant C(t)\left(\|\varphi\|_{L_2(\Gamma)} + 2\|f\|_{L_{2,1}(\Gamma_t)}\right), \tag{11}$$

it means independent of N estimate

$$\|y^N\|_{2,\Gamma_t} \leqslant C \quad (C > 0).$$

The latter means: from the sequence $\left\{y^N\right\}_{N\geqslant 1}$ with limited totality elements y^N can be distinguish the subsequence $\left\{y^{N_k}\right\}_{k\geqslant 1}$, that converge weakly to certain element $y \in W^{1,0}(a,\Gamma_T)$ at a norm $W^{1,0}(\Gamma_T)$ ($\left\{y^{N_k}\right\}_{k\geqslant 1}$ converge weakly to y together with $\frac{\partial y^{N_k}}{\partial x}$ at a norm $L_2(\Gamma_T)$). As a result of the consequent reasoning become clear that the all sequence $\left\{y^N\right\}_{N\geqslant 1}$ is weakly converges to an element $y \in W^{1,0}(a,\Gamma_T)$ (so as $\|\cdot\|_{W^{1,0}(\Gamma_T)} \leqslant \|\cdot\|_{2,\Gamma_T}$). Element $y(x,t)$ is a weak solution (4), (5). This is established by multiplying the first relation (10) to the absolutely continuous on $[0,T]$ function $d_i(t)$ ($d_i(T) = 0$) and next by summing along $i = \overline{1,N}$ and integrating results along t away from 0 up to T. After integrating the first term by parts to t obtain identity

$$\begin{gathered}-\int\limits_{\Gamma_T} y^N(x,t)\tfrac{\partial Y(x,t)}{\partial t}dxdt + \ell_T(y^N,Y) = \int\limits_{\Gamma} y^N(x,0)Y(x,0)dx + \\ + \int\limits_{\Gamma_T} f(x,t)Y(x,t)dxdt \quad \left(Y(x,t) = \sum_{i=1}^{N} d_i(t)u_i(x)\right).\end{gathered} \tag{12}$$

The set Σ (Σ is the set of all functions $Y(x,t)$ with arbitrary coefficients $d_i(t)$ possessing the above property, and arbitrary natural N) densely into the subspace of functions belonging to $W^1(a,\Gamma_T)$ and equal to zero at $t = T$. This follows from the density of the set $\{u_n(x)\}_{n\geqslant 1}$ in $W_0^1(a,\Gamma)$ and properties $Y(x,t)$ (continuity $Y(x,t) \in \Sigma$ along $t \in [0,T]$, $Y(x,t) \in W_0^1(a,\Gamma)$ for each fixed $t \in [0,T]$ and $Y(x,T) = 0$) as follows. Fix in (12) function $Y(x,t) = Y^*(x,t) \in \Sigma$ $\left(Y^*(x,t) = \sum_{i=1}^{N^*} d_i^*(t)u_i(x)\right)$ and on selected above subsequence $\{N_k\}_{k\geqslant 1}$, corresponding subsequence $\left\{y^{N_k}\right\}_{k\geqslant 1}$, we pass to the limit, starting with numbers $N_k \geqslant N^*$. As a result we obtain be a integral ratio (6) for the limit function $y(x,t)$ when $\eta(x,t) = Y^*(x,t)$, and that means, by virtue of the density of many Σ in the subspace of functions belonging to $W^1(a,\Gamma_T)$ and equal to zero when $t = T$, $y(x,t)$ is a weak solution of initial-boundary value problem (4), (5) of $W^{1,0}(a,\Gamma_T)$. Note that linearity of problem (4), (5) and estimate (11) guarantee the uniqueness of weak solutions $y(x,t)$. The statement of the theorem is proved.

Turn to the study of the resulting in proving theorems 1 weak solutions $y(x,t) \in W^{1,0}(a,\Gamma_T)$: show that $y(x,t)$ belongs to space $V^{1,0}(a,\Gamma_T)$.

Theorem 2. *If the conditions of the theorem 1, then initial-boundary value problem* (4), (5) *weakly solvability in space* $V^{1,0}(a,\Gamma_T)$ *for any* $0 < T < \infty$.

P r o o f. For proof sufficiently to establish that every fixed $t \in [0,T]$ track of belonging to space $W^{1,0}(a,\Gamma_T)$ weak problem solving (4), (5) is an element $W_0^1(a,\Gamma)$ and continuously depends from t in norm $W_2^1(\Gamma)$ (that is in norm $L_2(\Gamma)$). The tool of analysis uses the generalized Fourier method with base $\{u_i(x)\}_{i\geqslant 1}$, ortonormalization in $L_2(\Gamma)$ and orthogonal in sense of the scalar product $[\cdot,\cdot]$ (statement of the lemma 2). For simplicity of presentation, several narrow down space $L_{2,1}(\Gamma_T)$, replacing it on $CL_{2,1}(\Gamma_T) \subset L_{2,1}(\Gamma_T)$ (is the space of functions of $L_{2,1}(\Gamma_T)$, continuous along t in norm of $L_2(\Gamma)$), when this $f(x,t) \in CL_{2,1}(\Gamma_T)$ (the latter is easy condition in applications). In other words the functions $f(x,t)$, summable on the variable $t \in (0,T)$ in norm of $L_2(\Gamma)$ are replaced functions $f(x,t)$, that are continuous-variable $t \in [0,T]$ in norm of $L_2(\Gamma)$).

Consider a series

$$y(x,t) = \sum_{i=1}^{\infty}\left(\varphi_i e^{-\lambda_i t} + \int_0^t f_i(\tau)e^{-\lambda_i(t-\tau)}d\tau\right)u_i(x) \tag{13}$$

and decompositions

$$\varphi(x) = \sum_{i=1}^{\infty}\varphi_i u_i(x), \quad \varphi_i = \int_\Gamma \varphi(x)u_i(x)dx,$$
$$f(x,t) = \sum_{i=1}^{\infty} f_i(t)u_i(x), \quad f_i(t) = \int_\Gamma f(x,t)u_i(x)dx,\ t \in [0,T].$$

It is easy to verify that any from a finite segments of series (13) is a weak solution of problem (4), (5) that satisfy (6), boundary and initial conditions (5) (formally, ignoring the convergence). Further study of the character of the convergence of series (13) based on the properties of many eigenvalues and systems of generalized eigenfunctions, submitted statements of lemma 2, as well as on the analysis of norms $\|u(\cdot,t)\|_{W_2^1(\Gamma)}$, $\|u_t(\cdot,t)\|_{W_2^1(\Gamma)}$, $\int_0^t \|u_t(\cdot,t)\|^2_{W_2^1(\Gamma)}dt$, $\|u(\cdot,t)\|_{[\cdot,\cdot]}$, $\int_0^t \|u(\cdot,t)\|^2_{[\cdot,\cdot]}dt$ of series (13) [4] (see also [12, pp. 180, 185]). The latter represent a series of uniformly convergent about $t \in [0,T]$. Where should that the sum $y(x,t)$ of the series (13) belongs to $V^{1,0}(a,\Gamma_T)$ and is a weak solution (4), (5) for any T. Checking integral identity of definition 2 is similar to how it was done in proving the theorem 1: partially sum $y^N(x,t) = \sum_{i=1}^{N}\left(\varphi_i e^{-\lambda_i t} + \int_0^t f_i(\tau)e^{-\lambda_i(t-\tau)}d\tau\right)u_i(x)$ of series (13) is substituted in the integral identities (10) and conversions are performed, resulting in (12). Following passage to the limit under $N \to \infty$ conclude the reasoning. This means that the entire sequence $\left\{y^N\right\}_{N\geqslant 1}$ is weakly converges to an element (see proof of theorem 1).

Remark 4. When improvement the property of function $f(x,t)$ along variable t ($f(x,t) \in CL_{2,1}(\Gamma_T)$) raise the level of smoothness by $y(x,t)$ along $t > 0$ (compare under $t = 0$ $y(x,0) \in L_2(\Gamma)$). This stipulate to differentiability $y(x,t)$ under t when $t > 0$, as is characteristically for equations of parabolic type [4] (see also [12, p. 184]).

Theorem 3. *Initial-boundary value problem* (4), (5) *has only a weak solution in the space* $V^{1,0}(a,\Gamma_T)$ *for any* $0 < T < \infty$.

P r o o f. Proof of uniqueness by virtue of linearity problem (4), (5) is the standard way: assumes the existence of two different solutions $y_1(x,t)$, $y_2(x,t)$ of class $V^{1,0}(a,\Gamma_T)$ and shows (see (11) and the consequent reasoning) correctness of inequality $\|y\|_{2,\Gamma_T} \leqslant 0$ $(y(x,t) = y_1(x,t) - y_2(x,t))$ for any $T > 0$, and that means, coincidence solutions $y_1(x,t)$, $y_2(x,t)$ in space $V^{1,0}(a,\Gamma_T)$ ($y_1(x,t)$ similarly $y_2(x,t)$ almost everywhere with a fixed $t \in [0,T]$ for any $T > 0$).

Corollary 1. A weak solution of initial-boundary value problem (4), (5) continuously depends on the source data $f(x,t)$ and $\varphi(x)$. Thus, shows the correctness of Hadamard initial-boundary value problem (4), (5) in the space $V^{1,0}(a,\Gamma_T)$ for any $0 < T < \infty$.

Remark 5. Statements of theorems 1–3 are preserved under substitution $[0,T]$ on $[t_0,T]$ $(t_0 > 0)$, the initial condition in the ratio (5) is replaced by $y\mid_{t=t_0} = \varphi(x)$.

Remark 6. Boundary condition in (5) can be non-homogeneity:

$$y(x,t)\mid_{x\in\partial\Gamma} = \phi(x,t).$$

Proof of theorems in this case literally repeat the above reasonings. For this as a preliminary introduces a new unknown function $\widetilde{y}(x,t) = y(x,t) - \Phi(x,t)$ (here $\Phi(x,t)$ is a arbitrary function of $L_2(\Gamma_T)$, having generalized derivative $\frac{\partial\Phi}{\partial x} \in L_2(\Gamma_T)$ and satisfying (almost everywhere) non-homogeneity boundary condition (5)). The integral identities in definitions 1 and 2 be changed respectively.

In many applications where it is necessary to investigate the properties of solutions $y(x,t) \in V^{1,0}(a,\Gamma_T)$ of problem (4), (5) for an arbitrary finitely T, it is important to know the behavior $y(x,t)$ under $t \to +\infty$, i. e. when $t \in [0,\infty)$.

Let, as above $f(x,t) \in CL_{2,1}(\Gamma_T)$, and

$$\int\limits_t^{t+1} \|f(\cdot,\varsigma)\|^2_{L_2(\Gamma)} d\varsigma \leqslant A \tag{14}$$

for any $t \geqslant 0$ (A is fixed constant); the latter means that the function $f(x,t)$ is defined in the domain $\Gamma \times [0,T)$ for any finite T.

Theorem 4. *Let $y(x,t) \in V^{1,0}(a,\Gamma_T)$ is a weak solution of problem* (4), (5) *for an arbitrary $T > 0$ which $f(x,t)$, satisfies the condition* (14). *Then there is such positive constant C, that*

a) $\int\limits_t^{t+1} \|y(\cdot,\varsigma)\|^2_{W_2^1(\Gamma)} d\varsigma \leqslant C$ *for any* $t \geqslant 0$,

b) $\|y(\cdot,t))\|_{L_2(\Gamma)} \leqslant C$, *when* $t \to +\infty$.

P r o o f. In the conditions of theorems 2 and 3 the mapping $t \to y(x,t)$ $(t \geqslant 0)$ is continuously. Split half-line $[0,\infty)$ into sections $[j-1,j]$, $j = 1,2,...,$ and relabel t_j a number belonging to $[j-1,j]$, for which

$$\|y(\cdot,t_j)\|^2_{L_2(\Gamma_T)} = \max_{t\in[j-1,j]} \|y(\cdot,t)\|^2_{L_2(\Gamma_T)}, \quad j = 1,2,\dots. \tag{15}$$

For arbitrary positive s and t $(s < t)$ from the integral identities of the definition 1 of weak problem solving (4), (5) should the ratio

$$\int\limits_s^t \left(\frac{\partial y(x,t)}{\partial t}, y(x,t)\right)dt + \int\limits_s^t\int\limits_\Gamma \left(a(x)\frac{\partial y(x,t)}{\partial x}\frac{\partial y(x,t)}{\partial x} + b(x)y(x,t)y(x,t)\right) dxdt =$$
$$= \int\limits_s^t (f(x,t), y(x,t))dt,$$

if you put $\eta(x,t) = y(x,t)$ (ipso the remark 4 of theorem 3 the function $y(x,t)$ differentiable along t, $\frac{\partial y(x,t)}{\partial t} \in L_2(\Gamma_T)$). Of this ratio is obtained the inequality

$$\begin{gathered}\tfrac{1}{2}\|y(\cdot,t)\|^2_{L_2(\Gamma)} - \tfrac{1}{2}\|y(\cdot,s)\|^2_{L_2(\Gamma)} + \alpha \int\limits_s^t \|y(\cdot,\varsigma)\|^2_{W_2^1(\Gamma)} d\varsigma \leqslant \\ \leqslant \left(\int\limits_s^t \|f(\cdot,t)\|^2_{L_2(\Gamma)} dt\right)^{1/2} \left(\int\limits_s^t \|y(\cdot,\varsigma)\|^2_{W_2^1(\Gamma)} d\varsigma\right)^{1/2},\end{gathered} \tag{16}$$

where the positive constant α depends only on fixed a_* and β (see the condition (2)).

Further reasoning relies on the idea presented in the monograph by J.-L. Lions [13, p. 519]: use the inequality (16) for an arbitrary segment $[t_j, t_{j+2}]$, $j = 1, 2, ...$, because when condition (15) makes it possible the equality $t_j = t_{j+1}$.

We show that for any $j = 1, 2, ...$

$$\|y(\cdot, t_{j+2})\|_{L_2(\Gamma)} \leqslant \max\{\|y(\cdot, t_j)\|_{L_2(\Gamma)}, M\}, \tag{17}$$

where $M = \left(\frac{3A}{\alpha^2}\chi^2 + \frac{6A}{\alpha}\right)^{1/2}$, χ is the inclusion constant the space $W_2^1(\Gamma)$ in $L_2(\Gamma)$: $\|\nu\|_{L_2(\Gamma)} \leqslant \|\nu\|_{W_2^1(\Gamma)}$ for any $\nu \in W_2^1(\Gamma)$.

Let $\|y(\cdot, t_{j+2})\|_{L_2(\Gamma)} \leqslant \|y(\cdot, t_j)\|_{L_2(\Gamma)}$, then the proof is complete.

Suppose that

$$\|y(\cdot, t_{j+2})\|_{L_2(\Gamma)} > \|y(\cdot, t_j)\|_{L_2(\Gamma)}. \tag{18}$$

By virtue of (16) for $s = t_j$, $t = t_{j+2}$ true the inequality

$$\begin{gathered}\tfrac{1}{2}\|y(\cdot,t_{j+2})\|^2_{L_2(\Gamma)} + \alpha \int\limits_{t_j}^{t_{j+2}} \|y(\cdot,\varsigma)\|^2_{W_2^1(\Gamma)} d\varsigma \leqslant \\ \leqslant \tfrac{1}{2}\|y(\cdot,t_j)\|^2_{L_2(\Gamma)} + \left(\int\limits_{t_j}^{t_{j+2}} \|f(\cdot,\varsigma)\|^2_{L_2(\Gamma)} d\varsigma\right)^{1/2} \left(\int\limits_{t_j}^{t_{j+2}} \|y(\cdot,\varsigma)\|^2_{W_2^1(\Gamma)} d\varsigma\right)^{1/2},\end{gathered} \tag{19}$$

whence, comparing (19) to (18) it should be

$$\alpha \left(\int\limits_{t_j}^{t_{j+2}} \|y(\cdot,\varsigma)\|^2_{W_2^1(\Gamma)} d\varsigma\right)^{1/2} < \left(\int\limits_{t_j}^{t_{j+2}} \|f(\cdot,\varsigma)\|^2_{L_2(\Gamma)} d\varsigma\right)^{1/2}. \tag{20}$$

Taking into account $t_{j+2} - t_j \leqslant 3$ and the ratio (14), we obtain $\int\limits_{t_j}^{t_{j+2}} \|f(\cdot,\varsigma)\|^2_{L_2(\Gamma)} d\varsigma \leqslant 3A$ and the evaluation (20) takes a resultant form

$$\int\limits_{t_j}^{t_{j+2}} \|y(\cdot,\varsigma)\|^2_{W_2^1(\Gamma)} d\varsigma < \tfrac{3A}{\alpha^2}. \tag{21}$$

Since $t_{j+2} - t_j \geqslant 1$, then exist $\tau \in [t_j, t_{j+2}]$ such that $\int\limits_{t_j}^{t_{j+2}} \|y(\cdot,\varsigma)\|^2_{W_2^1(\Gamma)} d\varsigma = (t_{j+2} - t_j)\|y(\cdot,\tau)\|^2_{W_2^1(\Gamma)} \geqslant \|y(\cdot,\tau)\|^2_{W_2^1(\Gamma)}$, it means, from the evaluation (21) it should be

$$\|y(\cdot,\tau)\|_{L_2(\Gamma)} \leqslant \chi\sqrt{\tfrac{3A}{\alpha^2}}. \tag{22}$$

Let right how $s=\tau$, $t=t_{j+2}$ in the ratio (16), then using (21), come to the inequality

$$\tfrac{1}{2}\|y(\cdot,t_{j+2})\|^2_{L_2(\Gamma)} \leqslant \tfrac{1}{2}\|y(\cdot,\tau)\|^2_{L_2(\Gamma)} + (3A)^{1/2}\left(\tfrac{3A}{\alpha^2}\right)^{1/2},$$

or, taking into account (22),

$$\tfrac{1}{2}\|y(\cdot,t_{j+2})\|^2_{L_2(\Gamma)} \leqslant \tfrac{1}{2}\left(\chi\sqrt{\tfrac{3A}{\alpha^2}}\right)^2 + (3A)^{1/2}\left(\tfrac{3A}{\alpha^2}\right)^{1/2} = \tfrac{3A}{\alpha^2}\chi^2 + \tfrac{3A}{\alpha}$$

or

$$\|y(\cdot,t_{j+2})\|^2_{L_2(\Gamma)} \leqslant M^2 = \tfrac{6A}{\alpha^2}\chi^2 + \tfrac{6A}{\alpha}.$$

From here and the assumption (18) it should be (17) and further to any $t>0$ $(j=1,2,...)$ the inequality $\|y(\cdot,t))\|_{L_2(\Gamma)} \leqslant \max\{\max\limits_{t\in[0,2]}\|y(\cdot,t)\|_{L_2(\Gamma)}, M\} = \theta$ is correctly.

Considering the inequality (16) in the interval $[t,t+1]$ and given the assessment to $\|y(\cdot t))\|_{L_2(\Gamma)}$, get

$$\alpha\int\limits_t^{t+1}\|y(\cdot,\varsigma)\|^2_{W_2^1(\Gamma)}d\varsigma \leqslant \tfrac{1}{2}\theta^2 + A^{1/2}\left(\int\limits_t^{t+1}\|y(\cdot,\varsigma)\|^2_{W_2^1(\Gamma)}d\varsigma\right)^{1/2} < \vartheta$$

(here $\vartheta = \frac{1}{2}\theta^2 + \frac{\sqrt{3}A}{\alpha}$).

Thus, the assertion of the theorem is proved ($C=\max(\theta,\vartheta/\alpha)$), the solution $y(x,t)$ defined in domain $\Gamma_\infty = \Gamma_0\times[0,\infty)$.

3. Stability of system (4). Suppose $0\leqslant b(x)\leqslant\beta$ for $x\in\Gamma$ then the eigenvalues λ_i, $i\geqslant 1$, are positive (remark 3). Consider the system (4) on the set $\Gamma_\infty=\Gamma_0\times(0,\infty)$. Relabel $\Gamma_{t_0,t}=\Gamma_0\times(t_0,t)$, $\partial\Gamma_{t_0,t}=\partial\Gamma\times(t_0,t)$ $(0<t_0<t<\infty)$, $\Gamma_{t_0,\infty}=\Gamma_0\times(t_0,\infty)$, $\partial\Gamma_{t_0,\infty}=\partial\Gamma\times(t_0,\infty)$; it is clear that $\Gamma_{t_0,t}\subset\Gamma_t$. As above $f(x,t)\in CL_{2,1}(\Gamma_T)$, moreover $\int\limits_t^{t+1}\|f(\cdot,\varsigma)\|^2_{L_2(\Gamma)}d\varsigma\leqslant A$ for any $t\geqslant 0$.

Let the state of system (4) describes the function $\overline{y}(x,t)\in V^{1,0}(a,\Gamma_{t_0,\infty})$, what is a weak solution of equation (4) in the domain $\Gamma_{t_0,\infty}$ with initial and boundary conditions

$$y\mid_{t=t_0}=\overline{\varphi}(x),\quad x\in\Gamma,\quad y\mid_{x\in\partial\Gamma_{t_0,\infty}}=0, \tag{23}$$

and function $y(x,t)\in V^{1,0}(a,\Gamma_{t_0,\infty})$ is a weak solution of equation (4) in the domain $\Gamma_{t_0,\infty}$ with initial and boundary conditions

$$y\mid_{t=t_0}=\varphi(x),\quad x\in\Gamma,\quad y\mid_{x\in\partial\Gamma_{t_0,\infty}}=0 \tag{24}$$

(initial-boundary value problem (4), (23) differs from problem (4), (24) the fact that the function $\overline{\varphi}(x)$ of first ratio (23) replaced by a different function $\varphi(x)$). The state $\overline{y}(x,t)$ of system (4) call perturbed, and $y(x,t)$ is non-perturbed. From the proof of the theorem 3 (presentation (13) of weak problem solving (4), (5)) implies that states $\overline{y}(x,t)$, $y(x,t)$ are defined in the domain $\Gamma_{t_0,\infty}$, satisfy the corresponding initial and boundary conditions (23), (24) and belong to the space $V^{1,0}(a,\Gamma_{t_0,\infty})$ for $f(x,t)\in CL_{2,1}(\Gamma_\infty)$.

Below we offer the notion of stability solution to initial-boundary value problem (4), (5) (stability of the undisturbed system (4)) what analog for the Lyapunov stability.

Definition 3. The non-perturbed system (4) is stability, if for any $t_0>0$ and $\epsilon>0$ executed such $\delta(t_0,\epsilon)>0$ that under $\|\varphi-\overline{\varphi}\|_{L_2(\Gamma)}<\delta(t_0,\epsilon)$ is $\|y(\cdot,t)-\overline{y}(\cdot,t)\|_{W^1(a,\Gamma)}<\epsilon$ under $t\geqslant t_0$, where $y(x,t)$ is the perturbed status of system (4).

Remark 7. Similarly, the definition of stability of the non-perturbed system (4) you can enter the definition of uniformly stability of the non-perturbed system (4) in the domain $\Gamma_{t_0,\infty}$.

Remark 8. For the system (4) you can enter the definition of asymptotic and exponential stability of the non-perturbed system (4) in the domain $\Gamma_{t_0,\infty}$.

Remark 9. By virtue of the linearity of the system (4) all definitions can be reformulated for zero solution (trivial solution) of system (4).

Theorem 5. *Let* $0 \leqslant b(x) \leqslant \beta$, $x \in \Gamma$, *then the non-perturbed system* (4) *in the domain* Γ_T *is stability.*

P r o o f. By virtue of the linearity of equation (4) function $\theta(x,t) = y(x,t) - \overline{y}(x,t)$ is element of space $V^{1,0}(a, \Gamma_{t_0,\infty})$ and a weak solution of initial-boundary value problem for homogeneous equation (4) ($f = 0$), satisfies the initial and boundary conditions

$$\theta\mid_{t=t_0} = \phi(x), \quad x \in \Gamma, \quad \theta\mid_{x\in\partial\Gamma_{t_0,\infty}} = 0, \tag{25}$$

where $\phi(x) = \varphi(x) - \overline{\varphi}(x)$. Following the reasoning of proof of theorems 1–3, the initial-boundary value problem (4), (25) is uniquely weakly solvable, its solution has form $\theta(x,t) = \sum\limits_{i=1}^{\infty} \phi_i e^{-\lambda_i t} u_i(x)$, $\phi_i = (\phi, u_i)$, and is limit weakly converging sequence $\left\{\theta^N\right\}_{N\geqslant 1}$ of approximations

$$\theta^N(x,t) = \sum_{i=1}^{N} \phi_i e^{-\lambda_i t} u_i(x),$$

moreover

$$\|\theta^N\|_{2,\Gamma_t} \leqslant \sum_{i=1}^{N} \phi_i^2 e^{-2\lambda_i t}, \quad N = 1, 2, \dots .$$

Passage to the limit in the last inequality under $N \to \infty$ and bearing in mind the remark 3 we come to the estimate ($e^{-2\lambda_i t} < 1, \ \ i = 1, 2, \dots$)

$$\|\theta\|_{2,\Gamma_t} \leqslant C^* \left(\|\phi\|_{L_2(\Gamma)}\right)$$

for any $t \in [t_0, \infty)$; C^* is constant, not dependent on t. The latter means that (see (1))

$$\|\theta(\cdot,t)\|_{W^1(a,\Gamma)} \leqslant C^* \left(\|\phi\|_{L_2(\Gamma)}\right). \tag{26}$$

Fix $\epsilon > 0$, take $\delta = \frac{\epsilon}{C^*}$, then from (26) and

$$\|\phi\|_{L_2(\Gamma)} = \|\varphi - \overline{\varphi}\|_{L_2(\Gamma)} < \delta$$

it should be inequality $\|\theta(\cdot,t)\|_{W^1(a,\Gamma)} = \|y(\cdot,t) - \overline{y}(\cdot,t)\|_{W^1(a,\Gamma)} < \epsilon$ for any $t > t_0$. Proof of completed.

Example. Let Γ is graph-star with ribs γ_k, $k = 1,2,3$, and internal node ξ (to simplification the formulas assume that the ribs γ_k, $k = 1,2$, are parameterized of segment $[0, \pi/2]$, γ_3 is parameterized of segment $[\pi/2, \pi]$). In the domain $\Gamma \times [0,\infty)$ we consider the initial-boundary value problem (4), (5) under $a(x) = 1$, $b(x) = 0$ and $f(x,t) = 0$:

$$\tfrac{\partial y(x,t)}{\partial t} = \tfrac{\partial^2 y(x,t)}{\partial x^2}, \quad y\mid_{t=0} = \varphi(x), \quad x \in \Gamma, \quad y\mid_{x\in\partial\Gamma_T} = 0.$$

A weak solution $y(x,t) \in V^{1,0}(1, \Gamma_\infty)$ of a problem is determined by the identity

$$\int\limits_{\Gamma} y(x,t)\eta(x,t)dx - \int\limits_{\Gamma_t} y(x,t)\tfrac{\partial \eta(x,t)}{\partial t}dxdt + \int\limits_{\Gamma_t} \tfrac{\partial y(x,t)}{\partial t}\tfrac{\partial \eta(x,t)}{\partial t}dxdt = \\ = \int\limits_{\Gamma} \varphi(x)\eta(x,0)dx$$

for any $t \geqslant 0$ and any function $\eta(x,t) \in W^1(1,\Gamma_T)$. Easily to show [10, 11], that the spectral problem (7) ($a(x) = 1$, $b(x) = 0$) in the weak formulation (8) defines a set of eigenvalues $\{\lambda_n\}_{n\geqslant 1}$ ($\lambda_n = n^2$) and system of generalized eigenfunctions $\{u_n\}_{n\geqslant 1}$, where eigenvalues when $n = 2k-1$ is prime numbers, when $n = 2k$ have multiplicity 2, the corresponding generalized eigenfunctions are determined by the relations ($k = 1, 2, ...$)

$$u_{2k-1}(x) = \begin{cases} \cos(2k-1)(x-\frac{\pi}{2}), & x \in \gamma_1, \\ \cos(2k-1)(x-\frac{\pi}{2}), & x \in \gamma_2, \\ \cos(2k-1)(x-\frac{\pi}{2}), & x \in \gamma_3, \end{cases}$$

$$u_{2k,1}(x) = \begin{cases} \sin 2k(x-\frac{\pi}{2}), & x \in \gamma_1, \\ 0, & x \in \gamma_2, \\ \sin 2k(x-\frac{\pi}{2}), & x \in \gamma_3, \end{cases} \qquad u_{2k,2}(x) = \begin{cases} 0, & x \in \gamma_1, \\ \sin 2k(x-\frac{\pi}{2}), & x \in \gamma_2, \\ \sin 2k(x-\frac{\pi}{2}), & x \in \gamma_3. \end{cases}$$

As above in proving theorem 3 forming approximation of solution $y^N(x,t) = \sum\limits_{n=1}^{N} \varphi_n e^{-n^2 t} u_i(x)$ weakly converging under $N \to \infty$ to element $y(x,t) = \sum\limits_{n=1}^{\infty} \varphi_n e^{-n^2 t} u_n(x)$, $\varphi_n = (\varphi, u_n)$. Stability of the weak solution is obvious.

The work examined the initial-boundary value problem with Dirichlet boundary conditions in the ratios of (5). All estimates, reasoning and statements of theorem 1–5 persist for boundary conditions

$$(a(x)\frac{\partial y}{\partial x} + \sigma y) \mid_{x \in \partial\Gamma_T} = 0,$$

where σ is a positive constant.

4. Conclusion. The approach presented by the statements of theorem 3–5 may be used to obtain conditions of stability (asymptotic stability) of weak solution of initial-boundary value problem (4), (5). The same approach applies to both tasks, space variable x which has a dimension of large 1 ($x \in \mathbb{R}^n$, $n \geqslant 2$) and for many-dimensional functions that describe the state of researched system [14–16]. The results are fundamental in tasks of optimal control and stabilization of differential systems with time-delay [17–28].

References

1. Provotorov V. V., Gnilitskaya Yu. A. Granichnoe upravlenie volnovoi sistemoi v prostranstve obobshennyh reshenii na grafe [Boundary control of a wave systems in the space of the generalized solutions]. *Vestnik of Saint Petersburg University. Series 10. Applied Mathematics. Computer Science. Control Processes*, 2013, iss. 3, pp. 112–120. (In Russian)

2. Podvalny S. L., Provotorov V. V. Startovoe upravlenie parabolicheskoj sistemoj s raspredelennymi parametrami na grafe [Starting control of a parabolic system with distributed parameters on a graph]. *Vestnik of Saint Petersburg University. Series 10. Applied Mathematics. Computer Science. Control Processes*, 2015, iss. 3, pp. 126–142. (In Russian)

3. Provotorov V. V. Boundary control of a parabolic system with distributed parameters on a graph in the class of summable functions. *Automation and Remote Control*, 2015, vol. 76. iss. 2, pp. 318–322.

4. Provotorov V. V., Provotorova E. N. Sintez optimal'nogo granichnogo upravlenija parabolicheskoi sistemy s zapazdyvaniem i raspredelennymi parametrami na grafe [Synthesis of optimal boundary control of parabolic systems with delay and distributed parameters on the graph]. *Vestnik of Saint Petersburg University. Applied Mathematics. Computer Science. Control Processes*, 2017, vol. 13, iss. 2, pp. 209–224. https://doi.org/10.21638/11701/spbu10.2017.207 (In Russian)

5. Zhabko A. P., Tikhomirov O. G., Chizhova O. N. On stabilization of a class of systems with time proportional delay. *Vestnik of Saint Petersburg University. Applied Mathematics. Computer Science. Control Processes*, 2018, vol. 14, iss. 2, pp. 165–172. https://doi.org/10.21638/11702/spbu10.2018.209

6. Alexandrova I. V., Zhabko A. P. A new LKF approach to stability analysis of linear systems with uncertain delays. *Automatica*, 2018, vol. 91, pp. 173–178.

7. Provotorov V. V. Boundary control of a parabolic system with delay and distributed parameters on the graph. *International Conference "Stability and Control Processes" in memory of V. I. Zubov (SCP)*. Saint Petersburg, 2015, pp. 126–128.

8. Podvalny S. L., Provotorov V. V. The questions of controllability of a parabolic systems with distributed parameters on the graph. *International Conference "Stability and Control Processes" in memory of V. I. Zubov (SCP)*. Saint Petersburg, 2015, pp. 117–119.

9. Provotorov V. V., Volkova A. S. *Nachal'no-kraevie zadachi s raspredelennimi parametrami na grafe [Initial boundary value problems with distributed parameters on the graph]*. Voronezh, Nauchnay kniga Publ., 2014, 188 p. (In Russian)

10. Provotorov V. V. Sobstvennye funkcii zadachi Shturma—Liuvillja na grafe-zvezde [Eigenfunctions of the Sturm—Liouville problem on astar graph]. *Mathematics*, 2008, vol. 199, iss. 10, pp. 105–126. (In Russian)

11. Provotorov V. V. Razlozenie po sobstvennym funkciyam zadachi Shturma—Liuvillya na grafe-puchke [Expansion of eigenfunctions of Sturm—Liouville problem on bundle graph]. *Russian Mathematics*, 2008, vol. 3, pp. 50–62. (In Russian)

12. Ladyzhenskaya O. A. *Kraevye zadachi matematicheskoi fiziki [Boundary value problems of mathematical physics]*. Moscow, Nauka Publ., 1973, 407 p. (In Russian)

13. Lions J.-L. *Quelques methodes de resolution des problems aux limites non lineaire*. Paris, Dunod Gauther-Villars Press, 1969, 598 p. (Russ. ed.: Lions J.-L. *Nekotorye metody resheniya nelineinykh kraevykh zadach [Some methods of solving non-linear boundary value problems]*. Moscow, Mir Publ., 1972, 587 p.)

14. Provotorov V. V., Ryazhskikh V. I., Gnilitskaya Yu. A. Unique weak solvability of a nonlinear initial boundary value problem with distributed parameters in a netlike region. *Vestnik of Saint Petersburg University. Applied Mathematics. Computer Science. Control Processes*, 2017, vol. 13, iss. 3, pp. 264–277. https://doi.org/10.21638/11701/spbu10.2017.304

15. Provotorov V. V., Provotorova E. N. Optimal control of the linearized Navier—Stokes system in a netlike domain. *Vestnik of Saint Petersburg University. Applied Mathematics. Computer Science. Control Processes*, 2017, vol. 13, iss. 4, pp. 428–441. https://doi.org/10.21638/11701/spbu10.2017.409

16. Artemov M. A., Baranovskii E. S., Zhabko A. P., Provotorov V. V. On a 3D model of non-isothermal flows in a pipeline network. *Journal of Physics. Conference Series*, 2019, vol. 1203, Article ID 012094. https://doi.org/10.1088/1742-6596/1203/1/012094

17. Aleksandrov A. Yu., Zhabko A. P. O stabilizacii recheniya odnogo klassa nelineinyh differencial'nyh sistem [On stability of solutions to one class of nonlinear difference systems]. *Siberian Mathematical Journal*, 2003, vol. 44, no. 6, pp. 951–958. (In Russian)

18. Aleksandrov A., Aleksandrova E., Zhabko A. Asymptotic stability conditions for certain classes of mechanical systems with time delay. *WSEAS Transactions on Systems and Control*, 2014, vol. 9, pp. 388–397.

19. Aleksandrov A., Aleksandrova E., Zhabko A. Asymptotic stability conditions of solutions for nonlinear multiconnected time-delay systems. *Circuits Systems and Signal Processing*, 2016, vol. 35, no. 10, pp. 3531–3554.

20. Karelin V. V. Shtrafnye funkcii v zadache upravleniya processom nabludeniya [Penalty functions in the control problem of an observation process]. *Vestnik of Saint Petersburg University. Series 10. Applied Mathematics. Computer Science. Control Processes*, 2010, iss. 4, pp. 109–114. (In Russian)

21. Karelin V. V., Bure V. M., Svirkin M. V. Obobchennaja model rasprostranenija informacii v neprerivnom vremeni [The generalized model of information dissemination in continuous time]. *Vestnik of Saint Petersburg University. Applied Mathematics. Computer Science. Control Processes*, 2017, vol. 13, iss. 1, pp. 74–80. https://doi.org/10.21638/11701/spbu 10.2017.107 (In Russian)

22. Kamachkin A. M., Yevstafyeva V. V. Oscillations in a relay control system at an external disturbance. *Control Applications of Optimization 2000. Proceedings of the 11th IFAC Workshop*, 2000, vol. 2, pp. 459–462.

23. Veremey E. I., Sotnikova M. V. Stabilizaciya plazmy na baze prognoza s ustoichivym lineinym priblizheniem [Plasma stabilization by prediction with stable linear approximation]. *Vestnik of Saint Petersburg University. Series 10. Applied Mathematics. Computer Science. Control Processes*, 2011, iss. 1, pp. 116–133. (In Russian)

24. Aliseiko A. N. Matricy Lyapunova dlya klassa sistem s eksponencial'nym yadrom [Lyapunov matrices for a class of systems with an exponential core]. *Vestnik of Saint Petersburg University. Applied Mathematics. Computer Science. Control Processes*, 2017, vol. 13, iss. 3, pp. 228–240. https://doi.org/10.21638/11701/spbu 10.2017.301 (In Russian)

25. Ponomarev A. A. Approksimaciya obratnoi svyazi v regulyatore "predictor-corrector" yavnoi funkcii [Approximation of feedback in the "predictor-corrector" regulator by an explicit function]. *Vestnik*

of Saint Petersburg University. Applied Mathematics. Computer Science. Control Processes, 2017, vol. 13, iss. 2, pp. 193–208. https://doi.org/10.21638/11701/spbu 10.2017.206 (In Russian)

26. Kuptsova S. E., Kuptsov S. Yu., Stepenko N. A. On the limiting behavior of a time-delay system's solution. *Vestnik of Saint Petersburg University. Applied Mathematics. Computer Science. Control Processes*, 2018, vol. 14, iss. 2, pp. 173–182. https://doi.org/10.21638/11702/spbu10.2018.2010

27. Krasnov S. V., Sergeev S. M., Mukhanova N. V., Grushkin A. N. Methodical forming business competencies for private label. *6th International Conference on Reliability, Infocom Technologies and Optimization (Trends and Future Directions), ICRITO*, 2017, pp. 569–574.

28. Provotorov V. V., Sergeev S. M., Part A. A. Solvability of hyperbolic systems with distributed parameters on the graph in the weak formulation. *Vestnik of Saint Petersburg University. Applied Mathematics. Computer Science. Control Processes*, 2019, vol. 15, iss. 1, pp. 107–117. https://doi.org/10.21638/11702/spbu10.2019.2003

Received: August 15, 2019.
Accepted: November 07, 2019.

Author's information:

Aleksei P. Zhabko — Dr. Sci. in Physics and Mathematics, Professor; zhabko.apmath.spbu@mail.ru

Andrey I. Shindyapin — Dr. Sci. in Physics and Mathematics, Professor; shindyapin.andrey@gmail.com

Vyacheslav V. Provotorov — Dr. Sci. in Physics and Mathematics, Professor; wwprov@mail.ru

Устойчивость слабого решения параболической системы с распределенными параметрами на графе

А. П. Жабко[1], *А. И. Шиндяпин*[2], *В. В. Провоторов*[3]

[1] Санкт-Петербургский государственный университет, Российская Федерация,
199034, Санкт-Петербург, Университетская наб., 7–9
[2] Университет Эдуардо Мондлане, Мозамбик,
3453, Мапуту, бульвар Джулиуса Ньерере, 1
[3] Воронежский государственный университет, Российская Федерация,
394006, Воронеж, Университетская пл., 1

Для цитирования: *Zhabko A. P., Shindyapin A. I., Provotorov V. V.* Stability of weak solutions of parabolic systems with distributed parameters on the graph // Вестник Санкт-Петербургского университета. Прикладная математика. Информатика. Процессы управления. 2019. Т. 15. Вып. 4. С. 457–471. https://doi.org/10.21638/11702/spbu10.2019.404

Поведение решения эволюционного уравнения при неограниченном увеличении значений временной переменной давно является объектом обсуждения в научных кругах. Для этого есть немало причин прикладного характера, когда начальные условия уравнения задаются с определенной погрешностью: как малые изменения начальных условий влияют на поведение решения при больших значениях времени. Используется классическое понимание устойчивости решения дифференциального уравнения или системы уравнений, предложенное в работах А. М. Ляпунова: решение устойчиво, если оно мало изменяется при малых возмущениях начального условия и для любого момента времени. Установлены условия устойчивости решения эволюционной параболической системы с распределенными параметрами на графе, описывающей процесс переноса сплошной среды в пространственной сети. Параболическая система рассматривается в слабой постановке: слабым решением системы является суммируемая функция, удовлетворяющая интегральному тождеству, которое определяет вариационную постановку для начально-краевой задачи. Выход за рамки классических (гладких) решений и обращение к слабым решениям задачи продиктованы желанием авторов не только описать более точно физическую сущность процессов переноса (это приобретает особенное значение при изучении динамики многофазовых сред (multiphase media)), но и указать пути анализа процессов переноса в многомерных сетеподобных областях. Используемый подход основывается на априорных оценках слабого решения и построении (метод Фаедо—Галеркина со специальным базисом — системой собственных функций эллиптического оператора параболического уравнения) слабо компактного семейства приближенных решений в выбранном пространстве состояний. Полученные результаты лежат в основе анализа задач оптимального управления дифференциальными системами с распределенными параметрами на графе, при этом выявлены интересные аналогии с многофазовыми задачами многомерной гидродинамики.

Ключевые слова: эволюционная система параболического типа, распределенные параметры на графе, слабое решение, устойчивость слабого решения.

Контактная информация:

Жабко Алексей Петрович — д-р физ.-мат. наук, проф.; zhabko.apmath.spbu@mail.ru

Шиндяпин Андрей Игоревич — д-р физ.-мат. наук, проф.; shindyapin.andrey@gmail.com

Провоторов Вячеслав Васильевич — д-р физ.-мат. наук, проф.; wwprov@mail.ru